THE ACOUSTIC BASIS OF THE DIATONIC SCALE

to which is appended

THE PYTHAGOREAN RATIOS OF THE DIATONIC SCALE

by David G. Jensen

VERSION 1.1 OF THIS DOCUMENT

The principle is simple; its application is complex, but not complicated..

A1. Music, originally and primarily vocal music, is among the defining activities of human behavior and experience. Its formative social role was expounded by the ancient Greek philosopher Plato (*ca* 427-347 BCE), whose concepts continue to permeate Western thought. It may be affirmed in this regard that singing is not merely an art, acquired and perfected under professional guidance, but a behavior, a social behavior, properly as integral in the conduct of human life as language. Like language, its practice consists not of a series of arbitrary conventions, but is governed by the capacities of the place (mouth) and manner (voice) of its production, and, in the case of singing, the nature of its medium (sound). The latter is the subject of the present essay.

NB: The chronology in the following is not intended to be comprehensive,
but any account of the subject requires reference to particular persons and events.

A02. Historically, music by definition consists of various series and combinations of sounds, which are differentiated by *pitch* (vibrations per second) and *interval* (the relative differnce between pitches, both horizontal (melody) and vertical (harmony). These sounds are generally drawn from sets of *notes* (sounds differentiated by interval). Such a set is a *scale.* The sequence of notes which comprise a scale, *degrees,* are numbered (second, third *etc*) from the tonic to the octave (§A06). A scale distinguished by particular intervals between its member notes is a *mode.* Pitch occurs in a theoretical continuum, although its selection is by no means random (§A20). On the other hand, the selection of intervals, despite occasional assertions to the contrary, is very much guided by nature.

A03. The science of these and the remaining properties of sound is *acoustics.*

A specific musical sound has the following attributes:

amplitude is loudness;

wavelength is the spatial distance between successive oscillations of pitch (propagated in air, and visually represented in diagrams), the speed of which constitutes

frequency; the higher the frequency, the shorter the wavelength. An acoustic

period is the time which elapses between the successive sound waves; the higher the pitch, the shorter the period.

A04. Rather than these characteristics of individual notes, however, the present essay concerns the tuning of notes in relation to each other, and the various *modes* which can be created by altering their sequence. It regards the precise fixing of tonal distances *between* the notes in the melody, rather than *harmony,* the simultaneous recital of parallel sequence(s) of notes above and/or below the melody.

A05. On the other hand, the acoustic science of *temperament,* the fixing of precise intervals between the notes, is far less difficult and obscure than it sometimes is presented to be. The visual charts and diagrams, *etc* at times appear to require an advanced knowledge of mathematics. A practical understanding of the subject, however, needs no more than command of basic arithmetic, including fractions and decimals, and elementary algebra (several variables), as will be shown in the following paragraphs.

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A06. The issue became apparent in the Ancient World in the design and tuning of certain musical instruments. The Greek philosopher Pythagoras (*ca* 570 -*ca* 495 BCE), Plato's predecessor and known as well for his other mathematical discoveries, is credited with discovering the underlying principle of tonality, which is a numeric ratio. The legend is that he made this discovery listening to hammers of varying sizes and weights in a blacksmith's shop. He soon deduced that dividing the length of a string on an instrument in half will raise its note an *octave* (the eighth note following, perceived as the next highest occurrence of the key note of the scale). Further experimentation revealed that on a string pressed at two-thirds of its length, the lower, longer segment produces the fifth note (the *dominant*) above the *tonic* (key note). It was then easily deduced that the longer segment of a string pressed at three-fourths its length sounds the fourth degree (the *subdominant*) above the tonic. These proportions,

2 : 1 = octave, 3 : 2 = fifth and 4 : 3 = fourth

are among the primary and best-known tonal concepts in music. The sum of the factors, 2, 3 and 4, equals 10, which led Pythagoras and his followers to hold that number in special esteem. By happy coincidence, 10 is also the *radix* (numerical base) of Western and most other mathematics, although this likely derived from 10 being the number of fingers, the original counting device.

A07. Western music is *diatonic,* meaning that the notes which comprise a scale consist of two intervals: *tones* (also whole-steps) and *semitones* (also half-steps). As its name suggests, a semitone consists of half (variable within a narrow range) the frequency ratio of a tone, which can be characterized as half its "distance." The diatonic scale consists of seven notes, five tones and two semitones, beginning on the tonic and concluding with the octave, on which resumes the same sequence of intervals within a higher range of pitch.

A08. The semitones occur between series of two and three consecutive tones, respectively. By custom, the lower of two notes separated by a half-step is labeled the semitone; when the semitone is flat (reduced by a half-step), of course, the half-step separates it from the preceding note. The position of the semitones within the numerical sequence of notes in a scale varies according to mode. In the familiar "major" (historically *Ionian*) mode, this sequence consists of:

A. tone tone semitone tone tone tone semitone octave

The semitones here occur between the third-fourth and the seventh-eighth degrees of the scale.

Contrast the above with the sequence of intervals in the equally familiar "minor" (*Aeolian*) mode:

B. tone semitone tone tone semitone tone tone octave

The semitones here are found between the second-third and fifth-sixth degrees of the scale.

A09. The absolute pitches accorded the successive notes of the diatonic scale are by successive capital letters of the alphabet. These letters correspond with the white keys of a keyboard. The half-step is found between B-C and E-F.

A. Major: C D E – F G A B – C'

B. Minor: A B – C D E – F G A'

A10. As mentioned, these letters represent absolute pitches, centered within the broader range comfortable to the human ear and voice. Claims to the contrary notwithstanding, the specific pitches of the scale are fixed by custom or negotiated convention. The pitches accorded the various notes of a scale are computed, using the Pythagorean ratios, from the frequency of the tonic. Given that the specific pitches are arbitrary, logic suggests another set of names for the notes which is not dependent on the frequencies assigned to those intervals.

A11. Such a set of names given the notes is its s*olmisation.* In Western music, this is the familiar series *do, re, mi, fa, sol, la,* and *ti* (resuming on the next and higher *do*). This system was devised *ca* 1025 by a Benedictine monk, Guido of Arezzo (Italy). The names were derived from the initial syllables of a Latin hymn, although they were slightly modified later for practical purposes. The major scale begins on *do,* and the minor scale on *la.* From this, it can be easily deduced that the semitones are *mi* and *ti,* separated by a half-step from their following notes, *fa* and *do* respectively. The intervals in the *do-re-mi* scale are fairly stable. The use of accidentals (sharps and flats), modulation to another key, *etc* were not really an issue until instrumental music became more complex in the later Middle Ages and the Renaissance. In vocal music, the only common accidentals were b-flat (lowered by a half-step) and, to a lesser degree, f-sharp (raised by a half-step). In fact, the flat sign (♭) was in origin a stylized ‘b.’

A12. Intonation (§A27) begins with the relative pitches assigned by the ear to the various intervals, both melody (horizontal) and harmony (vertical). Although variable within a narrow range, the pitches of the different notes can be derived from the Pythagorean ratios described in §A06. This does not require that the singer inwardly calculate these ratios before each note. The Pythagorean ratios simply describe how the orderly structure of pitch differences within a piece of music is generated, a process which the ear is adept at discerning. This order is part of beauty. A melody is attractive not so much because of the sounds themselves, but the pattern of differences between the set of frequencies it contains. The other dynamic and complementary component of the process is the *overtone,* (also “harmonic”) *series* (§A22).

A13. The Pythagorean ratios for the three “perfect” intervals, (four, actually, including the tonic), the most important notes of the scale, were cited in §A06. The next step is to compute the ratios for the remaining notes of the diatonic scale. To do so comfortably requires the additional concept of *inversion.* An inversion of any interval is the same number of (half-)steps above the tonic counted down from the octave, which itself is the inversion of the tonic. Simply reversing the ratio of an interval provides its inversion, albeit in the precedinging octave. Thus, the inversion of *sol* (3 : 2) is 2 : 3, whcih produces *FA* (upper case in this context indicates the octave below the tonic). Multiply by 2 (see §A14, following) to obtain *fa* (see P06)*,* which is the invesion of *sol.*

A14. Likewise, if 2 : 1 indicates that the frequency of octave is twice that of the tonic. so 1 : 2 lowers it by the same degree. In functional terms, this means to multiply the octave times 1/2 to obtain the frequency of the tonic; and to multiply the tonic times 2/1 to obtain that of the octave. Thus, 2/3^{x} 2/1 does indeed obtain the Pythagorean ratio 4/3 for the perfect fourth. The intervals of fourth, fifth and octave are “perfect” because they alone (with a single, glaring exception, the augmented fourth in the Locrian mode, P06) are stable throughout the various modes, which turn out to be mere permutations of the diatonic scale (§A11).

A15. The calculation of Pythagorean intervals can be performed among the ratios alone, without reference to a specific frequency. These calculations require only the four arithmetic operations applied to fractions. The primary operations are multiplication (for an interval above) and division (for an interval below. For example the addition of a fourth above the fifth can be obtained as follows:

A. A fifth, 3/2 ^{x} a fourth, 4/3 = 3 ^{x} 4/2 ^{x} 3 = 12/6, or 2/1, or the Pythagorean ratio of the octave (§A06)

The inversion of this equation, a fifth below the octave, is obtained in this manner:

B. Using cross-multiplication, an octave, 2/1 ÷ a fifth, 3/2 = 2 ^{x} 2 / 1 ^{x} 3, or 4/3, the Pythagorean fourth.

A16. By using various combinations of these operations, it is possible to compute the Pythagorean ratios for all the degrees of the diatonic scale:

A. Major scale: *do* 3e0 : 2e0 *re* 3e2 : 2e3 *mi* 3e4 : 2e6 *fa* 2e2 : 3e1 *sol* 3e1 : 2e1 *la* 3e3 : 2e4 *ti* 3e5 : 2e7

B. Minor scale: *la* 3e0 : 2e0 *ti* 3e2 : 2e3 *do* 2e5 : 3e3 *re* 2e2 : 3e1 *mi* 3e1 : 2e1 *fa* 2e7 : 3e4 *sol* 2e4 : 3e2

A17. The only terms employed in Pythagorean ratios are exponentials of the integers 3 and 2 (3e0 and 2e0 both = 1), 3 usually the antecedent and 2 the consequent (3Ex : 2Ey), wtih some (§A17) apparent inversions, *e g* 2e2 : 3e1 in the Ionian mode. The reason for this is simple. Every degree of the chromatic (§A24) scale is distinquished by successive exponents of 3 (3e1 through 3e12) according to its position in the cycle of fifths. (P11). The term 2 in the ratio serves to locate the interval in the octave (in, above or below either the tonic or its octave (§A06).

A18. Despite certain current attempts to identify key with frequency, their correspondence is not absolute, and has varied historically and geographically. The current standard A4=440Hz is historically recent, and was not accepted internationally until 1955. This puts C4 (Middle C) at 261.656 Hz. The older standard, however, has some basis in nature, predicaticated on Pythagoras’ (§A06) teaching that all bodies (physical entities) emit vibrations analogous to musical frequencies, a phenomenon traditionally characterized as the “music of the spheres.”

A19. Modern science has affirmed that this assertion is true. It was predicted by the physicist Winfried Otto Schumann during 1952-4, and accurately measured a decade later, that the planet Earth emits a frequency of *ca* 8 Hz (7.83 Hz to be exact, although it varies slightly). Raising this frequency by five octaves (2e3 ^{x} 2e5 = 2e8) puts C4 at 256 Hz and A4 at 432 Hz, which in fact agrees with prior usage (§A02), proposed as a universal standard as early as 1713, and evidence for which also eremains in the structure of ancient instruments constructed *lit* millennia before (§A27). A4=432Hz has modern adherents for metaphysical as well as historical reasons, but is employed here (§A20) solely to reflect traditional use. Some find the lower frequencies preferable.aesthetically.

A20. A set of Tables under the title *Pythagorean Ratios of the Diatonic Scale* is appended to this essay (pp. 6-10), and includes the frequencies for each interval reckoned according to A4=432 (§A19).

021. Thus far, the system of Pythagorean ratios and their corresponding frequencies would appear to be compact and symmetrical, and diatonic tonality a closed system and consonant, *i e* free of unintended discord. Unfortunately, this is not the case, – or perhaps fortunately, for if it were, music would be less interesting, and far less capable of beauty. The culprit is the overtone series (also *harmonics,* or *partials,* §A12).

A22. The overtones are a set of less conspicuous frequencies which accompany the articulation (production) of any musical tone. The harmonic (overtone) series is generated simply by multiplying *f* (the *fundamental,* or tonic) ^{x} 1, 2, 3… . *f* ^{x} 1 is the first harmonic, ^{x} 2 the second harmonic, and so on. The harmonic series becomes progressively fainter and less audible in its successive multiples.

A23. It is sometimes claimed that diatonic tonality is the product of the harmonic series. This is partly true, in a sense complementary with the Pythagorean ratios, which govern the precise intervals. The problem is that the Pythagorean series is *exponential,* whereas the harmonic series is *arithmétic, i e* the Pythagorean frequencies are derived from various ratios of 3 and 2 (3Ex : 2Ey, and *vv.)*,whereas the harmonic series, as mentioned, is composed of successive multiples of *f* (^{x} 1, 2, 3, *etc*). Thus, their products do not correspond directly, but coincide only on increasingly distant, although predictable, occasions (P13). Historically, this became evident in the following manner:

A24. The primary harmonic interval (third in the harmonic series) is the fifth (Pythagorean 3 : 2). Multiplying this ratio by itself produces a series of twelve notes, each a semitone, which comprise the *chromatic* scale, culminating on the octave:

F C G D A E B F# C# G# D# A# E# (= F) [B# (= C')]

This series, call the *Circle,* or *Cycle, of Fifths,* extends seven octaves, and so 2e7 must be factored into 2e12 : 3e12 to reduce the range to the equivalent interval. Thus, 2e19 : 3e12 = 524,288 : 531,441

531,441 / 524,288 = 1.0136432 ; - 1.0000000 = .0136432 (Pythagorean comma)

The *Cycle of Fourths* is the inversion of the Cycle of Fifths:

B E A D G C F B♭ E♭ A♭ D# G♭ C♭ (= B) [F♭ (= E')]

NB: As in §A09, the apostrophe (')in this context denotes the octave above the tonic.

A25. Thus, the octave concluding the Cycle of Fifths is not exact, but is very slightly sharp. The difference is known as the *Pythagorean comma.* Such minor differences recur through the calculation of Pythagorean intervals , and are often also called different sorts of comma, these and more specific names having been assigned by Ancient Greek theorists. The reason is obvious: no power of 3, an odd integer, can equal any power of 2, an even integer. Historically, the problem is universal; it becomes evident in the design and tuning of musical instruments (§A06), in particular those requiring fixed. distances, such as the finger holes on winds, and the frets on stringed instruments (A contiguous issue, of course, is the frequency range of which an instrument is capable.) Cultural traditions have arrived at various solutions; all require some alteration, addition or omission of intervals in order to make the octave come out even and to obviate other incongruous dissonances. In the West, this question became acute in the design of keyboard instruments, first the organ in the Middle Ages, and especially during the Renaissance, with the development of those in which the strings are plucked (harpsichord) or struck (clavichord, and, later, piano), *etc*).

A26. The tuning problem became more difficult to avoid because the newer keyboards allowed more frequent accidentals (additional sharped and flatted notes), modulation, transposition and a broader range of keys (tonics) in general. This generated a period of unprecedented experimentation and development, in combination (*ensemble*) with other instruments as well. As the Pythagorean dissonances became more evident, systems of subtle alteration of pitches evolved in addition to those employed during the Middle Ages. As the present essay concerns vocal acoustics, this subject will not be discussed further, except to observe one compelling development (§A28) which remains an obstacle in the experience of older music, whether vocal or instrumental.

A27. The inherent acoustic asymmetry revealed by the Pythagorean ratios would have been evident, if not understood, in the earliest stagesof human musical culture. Artifacts excavated (§A20) in various regions include Neolithic-Era bone flutes, which require fixed intervals, over 40,000 years old. The issue was inescapable in the subsequent design of musical instruments. As also mentioned in §A25, different traditions adopted various solutions. The problem was most apparent in the definition and tuning of the semitones (*mi* and *ti*). At one extreme, China and neighboring cultures omitted them altogether. At the other extreme, tmusic in India, of similar antiquity, admitted and formalized microtones (intervals of less than a semitone). Every possible alternative, including the *shading* (nuanced variation in pitch according to context) of notes*,* is or has been employed at some place at some time. The tonality of Western Europe, as has been noted, is rooted in the mathematical speculations of Ancient Greece (§§A01 & A06).
The modification of ratios and frequencies to enable a true octave (§025) is *temperament,* although the term *intonation* is sometimes used in an analogous sense (§A12). Various resolutions of the Pythagorean dilemma have been attempted during the musical history of the West. Adjustments to this system were proposed as early as Aristoxenus (4th Century BCE) and Claudius Ptolemy (2nd Century CE). Much later (1722 & 1742), J. S. Bach's *Well-Tempered Clavier* promoted another example.

A28. A detailed history of temperament is beyond the scope of the present essay. Modern practice has accepted e*qual temperament* since the late Nineteenth Century, although this system had earlier advocates. Equal temperament resolves the problem by ignoring it, and reduces the diatonic scale to twelve equal semitones, each separated by 1.05946 Hz (= the 12th root of 2). This system has obvious advantages in the design of musical instruments, especially when mass production became possible with industrialization, and it facilitated the experiments in composition which ensued during the Twentieth Century, *i e* it radically changed Western tonality..

A29. Obviously, however, the intervals within the octave are subtly out of tune. It also distorts the enharmonic (§A23 & P13) effects of the overtone series. Moreover, it regards the degrees of the scale as functions of the octave, rather than *vice-versa* (§§17 &25). It is maintained that the advantages of the system far outweigh its deficits, and "accurate" intervals are defined as a narrow range of pitch which the ear is conditioned to perceive as equivalent. Indeed, the system and its various applications have been elevated to the virtual status of natural law. It does, however, require a preconscious re-tuning of auditory data to construct the natural tonal structure of what is heard,and the ear must be rigorously conditioned to hear equal-tempered intervals as "correct." Broadcast media are very effective at this. Even so, unaccompanied singer(s), by careful listening, may well recover, as some are, a very different experience, and one which is shared with their audience (§A01).

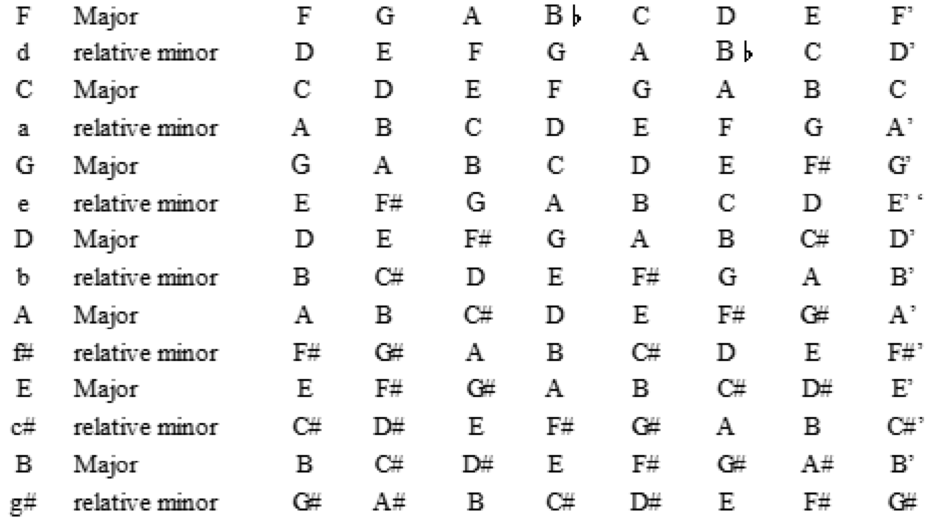
THE PYTHAGOREAN RATIOS OF THE DIATONIC SCALE

by David G. Jensen

P01. The diatonic scale consists ofseven notes: five tones (whole-steps) and two semitones (half-steps), which occur between two and three tones, respectively. In the major scale, the semitones are found between the third-fourth and seventh-eighth degrees, and in the minor scale, between the second-third and fifth-sixth.

P02. The *key signature* indicates those notes which must be raised (*sharps* #) or lowered (*flats* ♭) to form a major or minor scale on each successive note (as tonic) of the diatonic scale. The same key signature forms the minor (called *relative*) scale on the sixth degree of the major scale.

P03. TABLE I, below, shows the key signatures for the major and minor scales on each successive note (degree, not pitch) of the diatonic scale. The use of letters in the left-hand column differs slightly from those indicating the series of pitches in those keys. In the former context, capital letters indicate major keys, and lower-case represent minor keys.



F	Major	F	G	A	B ♭	C	D	E	F’
d	relative minor	D	E	F	G	A	B ♭	C	D’
C	Major	C	D	E	F	G	A	B	C
a	relative minor	A	B	C	D	E	F	G	A’
G	Major	G	A	B	C	D	E	F#	G’
e	relative minor	E	F#	G	A	B	C	D	E’ ‘
D	Major	D	E	F#	G	A	B	C#	D’
b	relative minor	B	C#	D	E	F#	G	A	B’
A	Major	A	B	C#	D	E	F#	G#	A’
f#	relative minor	F#	G#	A	B	C#	D	E	F#’
E	Major	E	F#	G#	A	B	C#	D#	E’
c#	relative minor	C#	D#	E	F#	G#	A	B	C#’
B	Major	B	C#	D#	E	F#	G#	A#	B’
g#	relative minor	G#	A#	B	C#	D#	E	F#	G#

P04. Other names, free of context, exist for degrees of the scale besides *do re mi*.

The following are comonly used for the notes of the diatonic scale:

1. tonic
2. supertonic
3. mediant
4. subdominan
5. dominan
6. submediant
7. leading tone
8. octave

The term *chromatic scale* refers to the twelve consecutive semitones contained within an octave:

1. first (tonic)
2. minor second
3. major second
4. minor third
5. major third
6. perfect fourth
7. augmented fourth*
8. perfect fifth
9. minor sixth
10. major sixth
11. minor seventh
12. major seventh

*The fourth is augmented in the Lydian mode (TABLE II), and the fifth is diminished in the Locrian. the two intervals are *enharmonic* (A4 = d5); see TABLE III. The interval is also called a *tritone,* as in it consists of three consectuive tones (whole steps).

P05. *Perfect* (abbrev. P) describes those intervals [first, fourth , fifth and octave) which, with two exceptions, rremain stable throughout the various modes; *minor* (m)denotes the lower of the two semitones which comprise a whole tone, and *major* (M) the higher of these semitones. *Augmentation* (A) is the rasing of any interval by a semitone (*e g* A6 = m7) and d*imunition* (d, adj. *diminished*) the lowering by a semitone (*e g* d4 = M3) . In the case of a whole tone which can be either major or minor, dimunition lowers it by a whole step, *e g* d6 = P5).

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PYTHAGOREAN RATIOS OF THE PYTHAGOREAN SCALE, cont.

P06. TABLE II. Below and on the following page are tabulated the Pythagorean ratios for the successive permutations of the diatonic scale. They extend here for three octaves, of Middle C (C4), the octave below (C3) and the octave above (C5). They comprise both the Authentic (tonic to octave) and Plagal, or "Hypo-" (5th below tonic to 5th above) modes (fourteen in all) on each degree of the scale. The notes constituting each scale are shown **bold type** in its column. The ratios in these charts employ scientific e-notation for exponents, *e g* 2e3 = 2 raised to the 3rd power; capital E is used when the exponent is a lower-case alphabetic variable, *e g* 2En). The upper-case letters (A, B, C *etc.*), however listed in ther *note* column here indicate relative rather than absolute pitches, as do the degrees..

degree	note	Ionian	Hypo-Ionian	Dorian	Hypo-Dorian	Phrygian	Hypo-Phrygian	Lydian	Hypo-Lydian
DO	C3	3e0 : 2e1	3e0 : 2e1	2e2 : 3e2	2e2 : 3e2	2e5 : 3e4	2e5 : 3e4	3e1 : 2e3	3e1 : 2e3
RE	D3	3e2 : 2e4	3e2 : 2e4	3e0 : 2e1	3e0 : 2e1	2e2 : 3e2	2e2 : 3e2	3e3 : 2e6	3e3 : 2e6
MI	E3	3e4 : 2e7	3e4 : 2e7	3e2 : 2e4	3e2 : 2e4	3e0 : 2e1	3e0 : 2e1	3e5 : 2e9	3e5 : 2e9
FA	F3	2e1 : 3e1	2e1 : 3e1	2e4 : 3e3	2e4 : 3e3	2e7 : 3e5	2e7 : 3e5	3e0 : 2e1	3e0 : 2e1
SOL	G3	3e1 : 2e2	**3e1 : 2e2**	2e1 : 3e1	2e1 : 3e1	2e4 : 3e3	2e4 : 3e3	3e2 : 2e4	3e2 : 2e4
LA	A3	3e3 : 2e5	**3e3 : 2e5**	3e1 : 2e2	**3e1 : 2e2**	2e1 : 3e1	2e1 : 3e1	3e4 : 2e7	3e4 : 2e7
TI	B3	3e5 : 2e8	**3e5 : 2e8**	3e3 : 2e5	**3e3 : 2e5**	3e1 : 2e2	**3e1 : 2e2**	3e6 : 2e10	3e6 : 2e10
do	C4	**3e0 : 2e0**	**3e0 : 2e0**	2e3 : 3e2	**2e3 : 3e2**	2e6 : 3e4	**2e6 : 3e4**	3e1 : 2e2	**3e1 : 2e2**
re	D4	**3e2 : 2e3**	**3e2 : 2e3**	**3e0 : 2e0**	**3e0 : 2e0**	2e3 : 3e2	**2e3 : 3e2**	3e3 : 2e5	**3e3 : 2e5**
mi	E4	**3e4 : 2e6**	**3e4 : 2e6**	**3e2 : 2e3**	**3e2 : 2e3**	**3e0 : 2e0**	**3e0 : 2e0**	3e5 : 2e8	**3e5 : 2e8**
fa	F4	**2e2 : 3e1**	**2e2 : 3e1**	**2e5 : 3e3**	**2e5 : 3e3**	**2e8 : 3e5**	**2e8 : 3e5**	**3e0 : 2e0**	**3e0 : 2e0**
sol	G4	**3e1 : 2e1**	**3e1 : 2e1**	**2e2 : 3e1**	**2e2 : 3e1**	**2e5 : 3e3**	**2e5 : 3e3**	**3e2 : 2e3**	**3e2 : 2e3**
la	A4	**3e3 : 2e4**	3e3 : 2e4	**3e1 : 2e1**	**3e1 : 2e1**	**2e2 : 3e1**	**2e2 : 3e1**	**3e4 : 2e6**	**3e4 : 2e6**
ti	B4	**3e5 : 2e7**	3e5 : 2e7	**3e3 : 2e4**	3e3 : 2e4	**3e1 : 2e1**	**3e1 : 2e1**	**3e6 : 2e9**	**3e6 : 2e9**
do'	C5	**2e1 : 3e0**	2e1 : 3e0	**2e4 : 3e2**	2e4 : 3e2	**2e7 : 3e4**	2e7 : 3e4	**3e1 : 2e1**	**3e1 : 2e1**
re'	D5	3e2 : 2e2	3e2 : 2e2	**2e1 : 3e0**	2e1 : 3e0	**2e4 : 3e2**	2e4 : 3e2	**3e3 : 2e4**	3e3 : 2e4
mi'	E5	3e4 : 2e5	3e4 : 2e5	3e2 : 2e2	3e2 : 2e2	**2e1 : 3e0**	2e1 : 3e0	**3e5 : 2e7**	3e5 : 2e7
fa'	F5	2e3 : 3e1	2e3 : 3e1	2e6 : 3e3	2e6 : 3e3	2e9 : 3e5	2e9 : 3e5	**2e1 : 3e0**	2e1 : 3e0
sol'	G5	3e1 : 2e0	3e1 : 2e0	2e3 : 3e1	2e3 : 3e1	2e6 : 3e3	2e6 : 3e3	3e2 : 2e2	3e2 : 2e2
la'	A5	3e3 : 2e3	3e3 : 2e3	3e1 : 2e0	3e1 : 2e0	2e3 : 3e1	2e3 : 3e1	3e4 : 2e5	3e4 : 2e5
ti'	B5	3e5 : 2e6	3e5 : 2e6	3e3 : 2e3	3e3 : 2e3	3e1 : 2e0	3e1 : 2e0	3e6 : 2e8	3e6 : 2e8
do''	C6	2e2 : 3e0	2e2 : 3e0	2e5 : 3e2	2e5 : 3e2	2e8 : 3e4	2e8 : 3e4	3e1 : 2e0	3e1 : 2e0

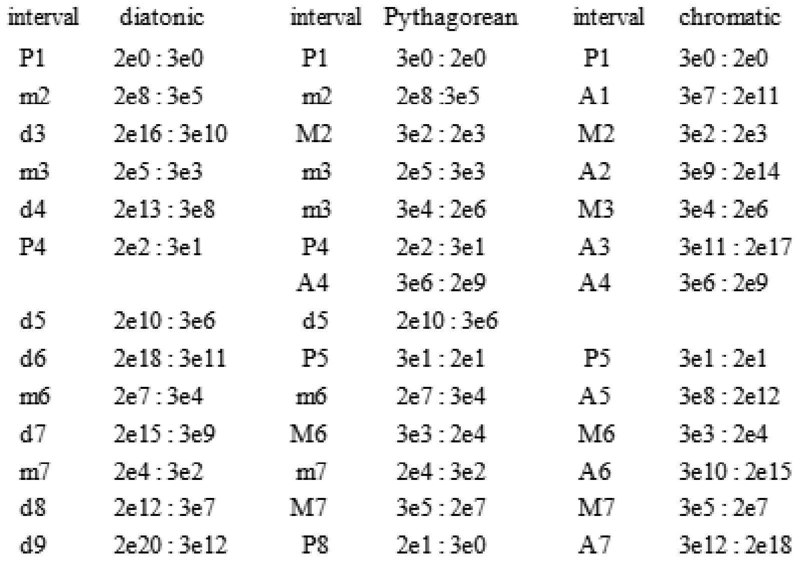
degree	note	Mixolydian	Hypo-Mixolydian	Aeolian	Hypo-Aeolian	Locrian	Hypo-Locrian
DO	C3	3e1 : 2e0	3e1 : 2e0	2e3 : 3e3	2e3 : 3e3	2e6 : 3e5	2e6 : 3e5
RE	D3	3e1 : 2e3	3e1 : 2e3	3e1 : 2e0	3e1 : 2e0	2e3 : 3e3	2e3 : 3e3
MI	E3	3e3 : 2e6	3e3 : 2e6	3e1 : 2e3	3e1 : 2e3	3e1 : 2e0	3e1 : 2e0
FA	F3	2e2 : 3e2	2e2 : 3e2	2e5 : 3e4	2e5 : 3e4	2e8 : 3e6	2e8 : 3e6
SOL	G3	2e1 : 3e0	2e1 : 3e0	2e2 : 3e2	2e2 : 3e2	2e5 : 3e4	2e5 : 3e4
LA	A3	3e2 : 2e4	3e2 : 2e4	2e1 : 3e0	2e1 : 3e0	2e2 : 3e2	2e2 : 3e2
TI	B3	3e4 : 2e7	3e4 : 2e7	3e2 : 2e4	3e2 : 2e4	2e1 : 3e0	2e1 : 3e0
do	C4	2e1 : 3e1	2e1 : 3e1	2e4 : 3e3	2e4 : 3e3	2e7 : 3e5	2e7 : 3e5
re	D4	3e1 : 2e2	**3e1 : 2e2**	2e1 : 3e1	2e1 : 3e1	2e4 : 3e3	2e4 : 3e3
mi	E4	3e3 : 2e5	**3e3 : 2e5**	3e1 : 2e2	**3e1 : 2e2**	2e1 : 3e1	2e1 : 3e1
fa	F4	2e3 : 3e2	**2e3 : 3e2**	2e6 : 3e4	**2e6 : 3e4**	2e9 : 3e6	**2e9 : 3e6**
sol	G4	**3e0 : 2e0**	**3e0 : 2e0**	2e3 : 3e2	**2e3 : 3e2**	2e6 : 3e4	**2e6 : 3e4**
la	A4	**3e2 : 2e3**	**3e2 : 2e3**	**3e0 : 2e0**	**3e0 : 2e0**	2e3 : 3e2	**2e3 : 3e2**
ti	B4	**3e4 : 2e6**	**3e4 : 2e6**	**3e2 : 2e3**	**3e2 : 2e3**	**3e0 : 2e0**	**3e0 : 2e0**
do'	C5	**2e2 : 3e1**	**2e2 : 3e1**	**2e5 :3e3**	**2e5 : 3e3**	**2e8 : 3e5**	**2e8 : 3e5**
re'	D5	**3e1 : 2e1**	**3e1 : 2e1**	**2e2 : 3e1**	**2e2 : 3e1**	**2e5 : 3e3**	**2e5 : 3e3**
mi'	E5	**3e3 : 2e4**	3e3 : 2e4	**3e1 : 2e1**	**3e1 : 2e1**	**2e2 : 3e1**	**2e2 : 3e1**
fa'	F5	**2e4 : 3e2**	2e4 : 3e2	**2e7 : 3e4**	2e7 : 3e4	**2e10 : 3e6**	**2e10 : 3e6**
sol'	G5	**2e1 : 3e0**	2e1 : 3e0	**2e4 : 3e2**	2e4 : 3e2	**2e7 : 3e4**	2e7 : 3e4
la'	A5	3e2 : 2e2	3e2 : 2e2	**2e1 : 3e0**	2e1 : 3e0	**2e4 : 3e2**	2e4 : 3e2
ti'	B5	3e4 : 2e5	3e4 : 2e5	3e2 : 2e2	3e2 : 2e2	**2e1 : 3e0**	2e1 : 3e0
do''	C6	2e3 : 3e1	2e3 : 3e1	2e6 : 3e3	2e6 : 3e3	2e9 : 3e5	2e9 : 3e5

PYTHAGOREAN RATIOS OF THE PYTHAGOREAN SCALE, cont.

P07. The degrees of the diatonic scale are derived from the cycle of fifths (§A24). Despite appearances, their culmination is not equal to the octave (§A25). Each successive fifth in the cycle is increasingly (albeit incrementally) sharp, just as each successive member of the cycle fourths (the inversion of the cycle of fifths) is increasingly flat. The sharp (#) of each pentatone (C, D, F, G, A) is theoretically *enharmonic* (equivalent in pitch to) with the flat (♭) of the note which follows. For the reason stated, this is not the case.

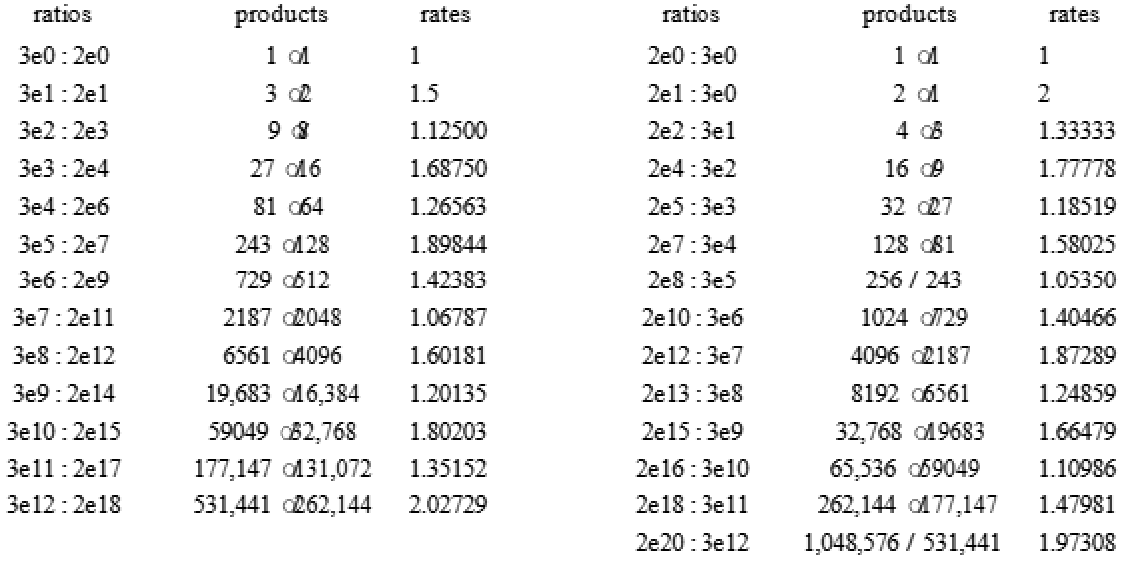
P08. TABLE III, below, in the center column shows the Pythagorean ratio for each degree ofthe chromatic scale computed from the three perfect ratios cited in (§A08). The column to the right shows these same intervals computed from the cycle of fifths; *chromatic* in this context refers exclusively to the semitones reckoned in this manner. Some are the same as the Pythagorean ratios. The column to the left represents the enharmonic intervals (called *diatonic* in this context) computed from the cycle of fourths. The remainder of the Pythagorean ratios are also found in this column. The diatonic series is the preferred method of reckoning semitones.

In the interval columns, P = perfect, m = minor, M = major, d= diminiished, and A = augmented.



interval	diatonic	interval	Pythagorean	interval	chromatic
P1	2e0 : 3e0	P1	3e0 : 2e0	P1	3e0 : 2e0
m2	2e8 : 3e5	m2	2e8 :3e5	A1	3e7 : 2e11
d3	2e16 : 3e10	M2	3e2 : 2e3	M2	3e2 : 2e3
m3	2e5 : 3e3	m3	2e5 : 3e3	A2	3e9 : 2e14
d4	2e13 : 3e8	m3	3e4 : 2e6	M3	3e4 : 2e6
P4	2e2 : 3e1	P4	2e2 : 3e1	A3	3e11 : 2e17
		A4	3e6 : 2e9	A4	3e6 : 2e9
d5	2e10 : 3e6	d5	2e10 : 3e6		
d6	2e18 : 3e11	P5	3e1 : 2e1	P5	3e1 : 2e1
m6	2e7 : 3e4	m6	2e7 : 3e4	A5	3e8 : 2e12
d7	2e15 : 3e9	M6	3e3 : 2e4	M6	3e3 : 2e4
m7	2e4 : 3e2	m7	2e4 : 3e2	A6	3e10 : 2e15
d8	2e12 : 3e7	M7	3e5 : 2e7	M7	3e5 : 2e7
d9	2e20 : 3e12	P8	2e1 : 3e0	A7	3e12 : 2e18

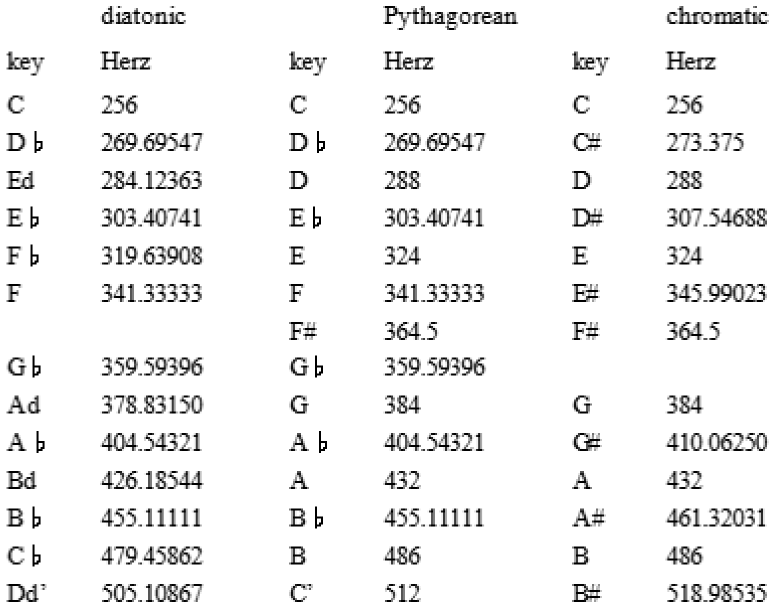
P09. TABLE IV, following, shows the entire set of ratios found in the present set of Tables, with their products represented as whole numbers, and their quotients of the fractions divided. The frequency of any note is determined by multiplying that of the *fundamental* (key note of the scale) by the quotient for that note; the term *rate,* in this context, is used rather than the term *quotient.*



ratios	products	rates	ratios	products	rates
3e0 : 2e0	1 / 1	1	2e0 : 3e0	1 / 1	1
3e1 : 2e1	3 / 2	1.5	2e1 : 3e0	2 / 1	2
3e2 : 2e3	9 / 8	1.12500	2e2 : 3e1	4 / 3	1.33333
3e3 : 2e4	27 / 16	1.68750	2e4 : 3e2	16 / 9	1.77778
3e4 : 2e6	81 / 64	1.26563	2e5 : 3e3	32 / 27	1.18519
3e5 : 2e7	243 / 128	1.89844	2e7 : 3e4	128 / 81	1.58025
3e6 : 2e9	729 / 512	1.42383	2e8 : 3e5	256 / 243	1.05350
3e7 : 2e11	2187 / 2048	1.06787	2e10 : 3e6	1024 / 729	1.40466
3e8 : 2e12	6561 / 4096	1.60181	2e12 : 3e7	4096 / 2187	1.87289
3e9 : 2e14	19,683 / 16,384	1.20135	2e13 : 3e8	8192 / 6561	1.24859
3e10 : 2e15	59049 / 32,768	1.80203	2e15 : 3e9	32,768 / 19683	1.66479
3e11 : 2e17	177,147 / 131,072	1.35152	2e16 : 3e10	65,536 / 59049	1.10986
3e12 : 2e18	531,441 / 262,144	2.02729	2e18 : 3e11	262,144 / 177,147	1.47981
			2e20 : 3e12	1,048,576 / 531,441	1.97308

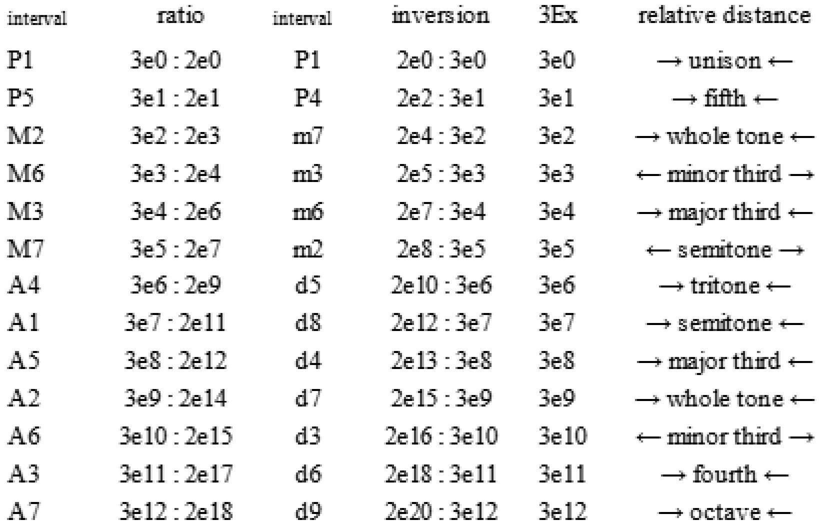
PYTHAGOREAN RATIOS OF THE PYTHAGOREAN SCALE, cont.

P10. TABLE V, below, shows the specific frequencies obtained from the ratios shown in TABLE IV, and which are computed from A4=432Hz by using the rates contained in that table. There are audible differences in the diatonic and chromatic frequencies for the ostensibly enharmonic intervals (*rf* §A25). Prior to the general accepance of equal temperament (§A28), these differences conferred a distinctive character on the different keys, which well into the Baroque Era (17th - 18th Centuries) actually required adjustments in the tuning of keyboard instruements during public recitals.



	diatonic		Pythagorean		chromatic
key	Herz	key	Herz	key	Herz
C	256	C	256	C	256
D ♭	269.69547	D ♭	269.69547	C#	273.375
Ed	284.12363	D	288	D	288
E ♭	303.40741	E ♭	303.40741	D#	307.54688
F ♭	319.63908	E	324	E	324
F	341.33333	F	341.33333	E#	345.99023
		F#	364.5	F#	364.5
G ♭	359.59396	G ♭	359.59396		
Ad	378.83150	G	384	G	384
A ♭	404.54321	A ♭	404.54321	G#	410.06250
Bd	426.18544	A	432	A	432
B ♭	455.11111	B ♭	455.11111	A#	461.32031
C ♭	479.45862	B	486	B	486
Dd'	505.10867	C'	512	B#	518.98535

P11. TABLE VI. Simple multiplication of 3e1 : 2e1 reveals that the exponent of the term 3 increases by 1 in each successive ratio in the cycle of fifths. It is also clear the the antecedent (numerator) 3Ex in each major interval corresponds to the consequent (denominator) in the minor interval which is its inverion. 3Ex in each instance represents the same acoustic distance, in the major instance up from the tonic, and in the minor down from the octave, as is shown in thecolumn on the right..

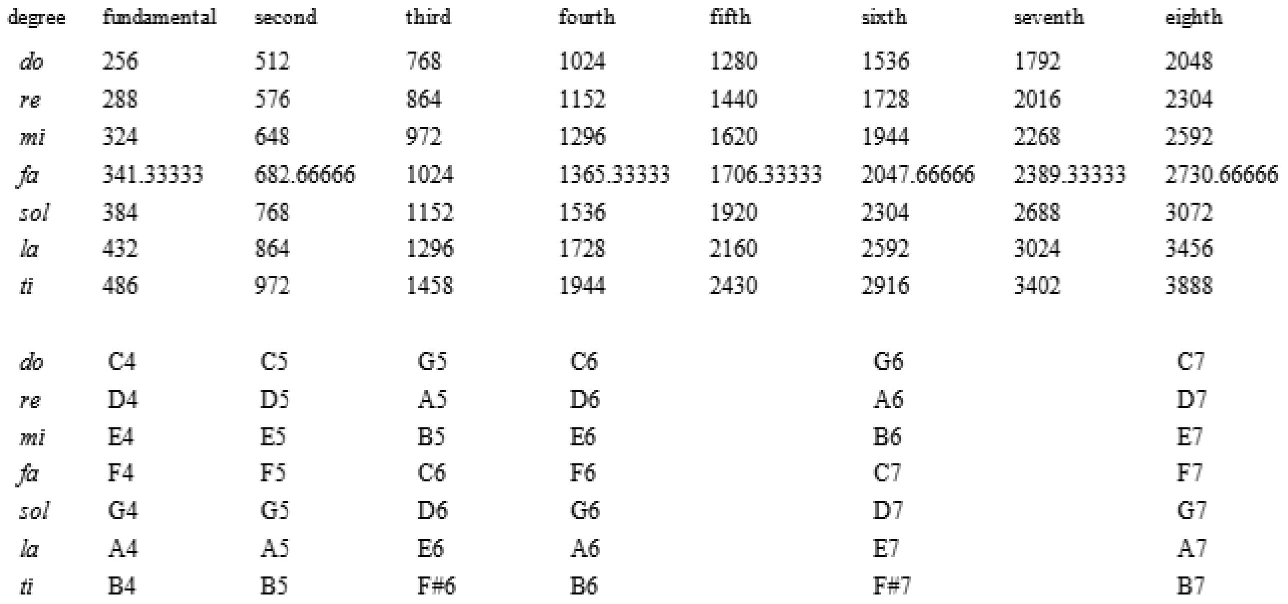


interval	ratio	interval	inversion	3Ex	relative distance
P1	3e0 : 2e0	P1	2e0 : 3e0	3e0	→ unison ←
P5	3e1 : 2e1	P4	2e2 : 3e1	3e1	→ fifth ←
M2	3e2 : 2e3	m7	2e4 : 3e2	3e2	→ whole tone ←
M6	3e3 : 2e4	m3	2e5 : 3e3	3e3	← minor third →
M3	3e4 : 2e6	m6	2e7 : 3e4	3e4	→ major third ←
M7	3e5 : 2e7	m2	2e8 : 3e5	3e5	← semitone →
A4	3e6 : 2e9	d5	2e10 : 3e6	3e6	→ tritone ←
A1	3e7 : 2e11	d8	2e12 : 3e7	3e7	→ semitone ←
A5	3e8 : 2e12	d4	2e13 : 3e8	3e8	→ major third ←
A2	3e9 : 2e14	d7	2e15 : 3e9	3e9	→ whole tone ←
A6	3e10 : 2e15	d3	2e16 : 3e10	3e10	← minor third →
A3	3e11 : 2e17	d6	2e18 : 3e11	3e11	→ fourth ←
A7	3e12 : 2e18	d9	2e20 : 3e12	3e12	→ octave ←

PYTHAGOREAN RATIOS OF THE PYTHAGOREAN SCALE, concl.

P12. TABLE VII, following, shows the first eight of the Overtone Series which accompany any musical sound. The Fundamental (original note) is counted as the first of the series. The harmonics (overtones) become progressively fainter, and are scarcely audible after the third overtone or so. The first eight are listed here for purposes of camparison.

P13. Directly below are shown the enharmonic notes of the scale (those occupying the same frequencies). The overtones serve to reinforce the identity of these notes and to correct the miniscule sonic discrepancies generated by the precise ratios which reflect (*lit.* resound) nature. That is to say, they assist the singer to "correct" the frequencies to conform with the acoustically exact intervals. This tendency of singers was attested as long ago as the Sixteenth Century by the Italian theorist Gioseffo Zarlino (1517 – 1590). The issue had become apparent long before with the development of musical instruments which permit only a single value (*e g* by use of frets) for each interval. On the other hand, players of unfretted instruments such as the cello constantly, albeit subtly, adjust the pitches of various intervals throughout a performance.



degree	fundamental	second	third	fourth	fifth	sixth	seventh	eighth
do	256	512	768	1024	1280	1536	1792	2048
re	288	576	864	1152	1440	1728	2016	2304
mi	324	648	972	1296	1620	1944	2268	2592
fa	341.33333	682.66666	1024	1365.33333	1706.33333	2047.66666	2389.33333	2730.66666
sol	384	768	1152	1536	1920	2304	2688	3072
la	432	864	1296	1728	2160	2592	3024	3456
ti	486	972	1458	1944	2430	2916	3402	3888
do	C4	C5	G5	C6		G6		C7
re	D4	D5	A5	D6		A6		D7
mi	E4	E5	B5	E6		B6		E7
fa	F4	F5	C6	F6		C7		F7
sol	G4	G5	D6	G6		D7		G7
la	A4	A5	E6	A6		E7		A7
ti	B4	B5	F#6	B6		F#7		B7

P14. CODA. In conclusion, it is appropriate to emphsize that the purpose of these Tables and the accompanying essay *The Acoustic Basis of the Diatonic Scale* is not to espouse any specific system of temperament, but to demonstrate the properties of sound which generate the diatonic scale. Just as the choice of instrument affects the pitch of various intervals, so likewise does the anatomy of the human voice. The data cited above describe sound produced in an isolated acoustic environment, *i e* without distortion in either production or place. Even so, the strength of the system and the tonal "ear" of those who participate generally compensate for any tonal ambiguity.